Generating Functions for Canonical Transformations

- We want to <u>construct</u> a canonical trafo such that

$$Q_i = Q_i(q_j, p_j, t), \quad P_i = P_i(q_j, p_j, t), \quad \underset{\substack{\uparrow \\ \text{Hamiltonian in new coordinates}}}{K} = K(Q_j, P_j, t)$$

with

$$\dot{Q}_j = \frac{\partial K}{\partial P_j}, \quad \dot{P}_j = -\frac{\partial K}{\partial Q_j}$$

(and $\{Q_j, P_k\} = \delta_{jk}$, $\{Q_j, Q_k\} = \{P_j, P_k\} = 0$)

Hamilton's equs. can be derived from the variational principle

$$\delta \int_{t_1}^{t_2} \Big(\sum_k p_k \dot{q}_k - H(q_j, p_j, t) \Big) dt = 0$$

which should be invariant under the canonical trafo as well, so we must also have (up to a total time derivative),

$$\delta \int_{t_1}^{t_2} \Big(\sum_k P_k \dot{Q}_k - K(Q_j, P_j, t) \Big) dt = 0$$

$$\Rightarrow \quad \sum_k p_k \dot{q}_k - H(q_j, p_j, t) = \sum_k P_k \dot{Q}_k - K(Q_j, P_j, t) + \frac{d\Phi}{dt}.$$

$$\Leftrightarrow \quad \boxed{\sum_k (p_k \, dq_k - P_k \, dQ_k) + (K - H)\, dt = d\Phi} \qquad (*)$$

Also defines a canonical trafo. Eq. (*) suggests that we view Φ as a function of the old and new generalized coordinates.

Suppose we can solve

$$Q_i = Q_i(q_j, p_j, t) \rightarrow p_i(q_j, Q_j, t) \qquad (**)$$

$$\hookrightarrow \quad P_i = P_i(q_j, p_j(q_j, Q_j, t), t)$$

$\Rightarrow$ (q_j, Q_j, t) are independent variables. Define

$$F_1 \equiv \Phi\left(q_j, p_j(q_j, Q_j, t), t\right) = F_1(q_j, Q_j, t)$$

$$\Rightarrow dF_1 = \sum_k \left(\frac{\partial F_1}{\partial q_k} dq_k + \frac{\partial F_1}{\partial Q_k} dQ_k\right) + \frac{\partial F_1}{\partial t} dt$$

$$\overset{(*)}{=} \sum_k \left(p_k\, dq_k - P_k\, dQ_k\right) + (K - H)\, dt$$

$$\Rightarrow \boxed{p_j = \frac{\partial F_1}{\partial q_j}, \quad P_j = -\frac{\partial F_1}{\partial Q_j}, \quad K(Q_j, P_j, t) = H(q_j(Q_j, P_j, t), p_j(Q_j, P_j, t), t) + \frac{\partial F_1}{\partial t}}$$

We assumed in Eq. (**) and the discussion that we can solve for $p_i(q_j, Q_j, t)$, but what if that is inconvenient or impossible, e.g., for the identity trafo

$$Q_j = q_j, \quad P_j = p_j \quad ?$$

$\Rightarrow$ Take (q_j, P_j, t) (or (Q_j, p_j, t)) as independent vars!

Express second term of Eq. (*) as

$$-\sum_k P_k\, dQ_k = -d\left(\sum_k P_k Q_k\right) + \sum_k Q_k\, dP_k$$

to obtain

$$\sum_k \left(p_k\, dq_k + Q_k\, dP_k\right) + (K - H)\, dt = d\left(\sum_k P_k Q_k + \Phi\right)$$

$$\Rightarrow F_2(q_j, P_j, t) \equiv \sum_k P_k Q_k(q_j, P_j, t) + \underbrace{\Phi(q_j, p_j(q_j, P_j, t), t)}_{\simeq \tilde{F}_1(q_j, P_j, t)}$$ Legendre Trafo!

$$\Rightarrow p_j = \frac{\partial F_2}{\partial q_j}, \quad Q_j = +\frac{\partial F}{\partial P_j}, \quad K(Q_j, P_j, t) = H(q_j, p_j, t) + \frac{\partial F_2}{\partial t}.$$

We can find four basic types of canonical transformations via Legendre trafos:

$$F_1 = \Phi(q_j, Q_j, t), \quad p_j = \frac{\partial F_1}{\partial q_j}, \quad P_j = -\frac{\partial F_1}{\partial Q_j}, \qquad K = H + \frac{\partial F_1}{\partial t}$$

$$F_2 = \sum_k P_k Q_k + \tilde{\Phi}(q_j, P_j, t), \quad p_j = \frac{\partial F_2}{\partial q_j}, \quad Q_j = \frac{\partial F_2}{\partial P_j}, \quad K = H + \frac{\partial F_2}{\partial t}$$

$$F_3 = -\sum_k q_k p_k + \tilde{\Phi}(p_j, Q_j, t), \quad q_j = -\frac{\partial F_3}{\partial p_j}, \quad P_j = -\frac{\partial F_3}{\partial Q_j}, \quad K = H + \frac{\partial F_3}{\partial t}$$

$$F_4 = -\sum_k (q_k p_k - Q_k P_k) + \check{\Phi}(p_j, P_j, t), \quad q_j = -\frac{\partial F_4}{\partial p_j}, \quad Q_j = \frac{\partial F_4}{\partial P_j}, \quad K = H + \frac{\partial F_4}{\partial t}$$

Can mix F_1, F_2, F_3, F_4 for pairs of canonical vars $(q, p) \to (Q, P)$ (e.g. transform (q_1, p_1) according to F_1, (q_2, p_2) according to $F_2, \dots$)

Examples : i) Application of generating functions

$$F_2(q_j, P_j, t) = \sum_{k=1}^{n} q_k P_k$$

We have

$$p_j = \frac{\partial F_2}{\partial q_j} = P_j \quad , \quad Q_j = \frac{\partial F_2}{\partial P_j} = q_j$$

which is just the identity.

ii)

$$F_1(q_j, Q_j, t) = \sum_{k=1}^{n} q_k Q_k$$

$$p_j = \frac{\partial F_1}{\partial q_j} = Q_j \quad , \quad P_j = -\frac{\partial F_1}{\partial Q_j} = -q_j$$

↳ Up to a sign, this is an interchange of coordinates and momenta.